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Topological Hochschild homology of algebras in characteristic p

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Abstract

We compute the topological Hochschild homology modules of finitely generated commutative algebras over finite fields provided their singularities are sufficiently mild. © 2000 Elsevier Science B.V. All rights reserved.

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0. Introduction

Topological Hochschild homology is a functor from the category of rings with unit to that of spectra, or alternatively, a sequence of functors

$$\mathrm{THH}_i = \pi_i^S(\mathrm{THH}(-)), \quad i \geq 0,$$

from the category of rings with unit to that of abelian groups. Viewed in the latter way, it arises naturally in several different contexts – as a spectrum analogue of the Hochschild homology construction in [3, 10], as stable K-theory [8], and as Mac Lane homology [27].

The first approach, that of constructing a spectrum analogue of the Hochschild homology complex, was suggested by T. Goodwillie and first carried out by M. Bökstedt. It provides an obvious linearization map from THH(A) to the Hochschild

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homology complex, and on homotopy groups it gives a collection of maps

$$\mathrm{THH}_*(A) \xrightarrow{\ell} \mathrm{HH}_*(A). \tag{0.1}$$

The Dennis trace map from algebraic K-theory to Hochschild homology factors through ℓ, and THH_* seems likewise intermediate in difficulty between HH_* and K_* from a computational standpoint. In this paper, we describe an approach to the problem of computing the topological Hochschild homology groups of commutative $\mathbb{F}_p$-algebras. Our starting point is the theorem of Bökstedt [4] and Breen [6] that

$$\mathrm{THH}_*(\mathbb{F}_p) \cong \mathbb{F}_p[u], \quad \deg u = 2. \tag{0.2}$$

We will show that for many unital $\mathbb{F}_p$-algebras A, THH *splits weakly*. This implies in particular that there is an isomorphism of graded A-modules

$$\mathrm{THH}_*(A) \cong \mathrm{HH}_*(A) \otimes \mathrm{THH}_*(\mathbb{F}_p). \tag{0.3}$$

In other words,

$$\mathrm{THH}_k(A) \cong \mathrm{HH}_k(A) \oplus \mathrm{HH}_{k-2}(A) \oplus \mathrm{HH}_{k-4}(A) \oplus \cdots.$$

Unfortunately, we cannot prove that the isomorphism (0.3) is canonical nor that it respects multiplicative structure. However, we do know that in terms of the decomposition (0.3), the linearization ℓ is the augmentation $u \mapsto 0$ in the second factor.

It is conceivable that $\mathrm{THH}_*(A)$ splits weakly, or even splits canonically, for every $\mathbb{F}_p$-algebra A, but there are recent examples [24, 25] which show that the analogous splitting principle for $\mathbb{Z}$-algebras and $\mathrm{THH}_*(\mathbb{Z})$ breaks down quite dramatically (with the linearization map ℓ becoming the zero map in high enough dimension).

Because topological Hochschild homology commutes with taking étale extensions, it suffices to analyse the problem of splitting étale-locally, which amounts to the same thing as working formally. We show that as long as all the singularities of the spectrum of a ring A are of a certain analytic type (*monoidal*, see Definition 3.1), THH splits weakly for A. The basic strategy is very similar to that used in [11] to compute the cyclic homology of seminormal curves.

It is difficult to say exactly how common monoidal singularities are in nature, but they arise fairly often and in several different contexts. Toric singularities are always monoidal, but monoidal singularities are more general, since they need not be normal. For example, all simple (or “A-D-E”) singularities of plane curves are monoidal, at least if $p > 5$. So are seminormal singularities of curves. Several other examples are given below.

The paper is organized as follows. In Section 1 we briefly develop the facts about Hochschild homology that we need below. In Section 2, we discuss topological Hochschild homology and establish some splitting criteria. The main theorem appears in Section 3, along with a list of examples to show that monoidal singularities are common.

Throughout this paper, A will denote a finitely generated commutative unital $\mathbb{F}_p$-algebra, and $\mathbb{N}$ the monoid of non-negative integers. Rings will always be assumed to be commutative and unital. Unless specified to the contrary, tensor products are taken over $\mathbb{F}_p$ and Hochschild homology is relative to $\mathbb{F}_p$.

1. Hochschild homology

We begin by gathering some useful facts about Hochschild homology of commutative finitely generated $\mathbb{F}_p$-algebras, most of which are more or less standard. We write A^e for $A\otimes A$ and regard A as an A^e-algebra via the multiplication homomorphism $a\otimes b\mapsto ab$.

Lemma 1.1. *For all $k\in\mathbb{N}$, $\mathrm{HH}_k(A)$ is a finitely generated A-module.*

Proof. As our base ring is a field, A is flat, so for all $k\in\mathbb{N}$,

$$\mathrm{HH}_k(A)=\mathrm{Tor}_k^{A^e}(A,A).$$

As A is finitely generated, the same is true for A^e, so the latter ring is Noetherian. In particular, any submodule of a finitely generated A^e-module is finitely generated. Therefore, A, as a finitely generated A^e-module, admits a resolution by finitely generated free A^e-modules. Tensoring this complex by the A^e-algebra A, we obtain a complex of finitely generated A-modules. It follows that $\mathrm{HH}_k(A)$ is a finitely generated A-module for all k. $\square$

Lemma 1.2. *If B is an étale A-algebra, for all $k\in\mathbb{N}$ the natural map*

$$\mathrm{HH}_k(A)\otimes_A B\xrightarrow{\sim}\mathrm{HH}_k(B) \tag{1.1}$$

is an isomorphism.

Proof. See [12, Theorem 0.1]. $\square$

Lemma 1.3. *If $\{A_\alpha\}_\alpha$ is a filtered system of finitely generated $\mathbb{F}_p$-algebras with direct limit A, then*

$$\varinjlim_\alpha \mathrm{HH}_k(A_\alpha)\xrightarrow{\sim}\mathrm{HH}_k(A)$$

is an isomorphism for all $k\in\mathbb{N}$.

Proof. Direct limits are exact and commute with tensor products. $\square$

Lemma 1.4. *If S is a multiplicative system in A, then*

$$\mathrm{HH}_k(A)\otimes_A S^{-1}A\xrightarrow{\sim}\mathrm{HH}_k(S^{-1}A)$$

is an isomorphism for all $k \in \mathbb{N}$. *If* $\mathfrak{p}$ *is a prime ideal of* A *then*

$$\mathrm{HH}_k(A) \otimes_A \tilde{A}_{\mathfrak{p}} \xrightarrow{\sim} \mathrm{HH}_k(\tilde{A}_{\mathfrak{p}})$$

is an isomorphism for all $k \in \mathbb{N}$, *where* $\tilde{A}_{\mathfrak{p}}$ *denotes the henselization of the local ring* $A_{\mathfrak{p}}$.

Proof. This is immediate from Lemmas 1.2 and 1.3. □

Definition 1.5. We say that a prime ideal $\mathfrak{p}$ of A is a *gradable* point if $\hat{A}_{\mathfrak{p}}$ is isomorphic to the completion of a graded ring R with respect to its augmentation ideal. We say that a ring A is *gradable* if every prime ideal of $\operatorname{Spec} A$ is gradable.

Lemma 1.6. *Every non-singular point of an affine variety is gradable.*

Proof. Any ring of power series is isomorphic to the completion of a polynomial ring, endowed with the usual grading. □

Proposition 1.7. *Let* $\mathbb{F}$ *be an algebraic extension of* $\mathbb{F}_p$ *and* $P(x,y) \in \mathbb{F}[x,y]$ *a polynomial in two variables. If* $A = \mathbb{F}[x,y]/(P(x,y))$ *is reduced and gradable, then*

$$\mathrm{HH}_n(A) = \begin{cases} A & n=0, \\ A^2/\{(\bar{P}_x a, \bar{P}_y a) \mid a \in A\} & n=1, \\ A/(\bar{P}_x A + \bar{P}_y A) & n \geq 2, \end{cases} \tag{1.2}$$

as A*-modules, where* $\bar{P}_x$ *and* $\bar{P}_y$ *denote partial derivatives of* $P \in \mathbb{F}[x,y]$, *regarded as elements of* A.

Proof. The formulae for HH_i for $i \leq 2$ are essentially due to Wolffhardt [28], as is the fact that $\mathrm{HH}_k(A) \cong \mathrm{HH}_{k+2}(A)$ for $k \geq 2$. The fact that, in the graded case, the even and odd Hochschild homology modules coincide is due to Michler [26]. □

Example 1.8. The even and odd Hochschild homology groups need not coincide as A-modules for general plane curves. Michler gives the example $x^3y^2 + y^5 + x^7 = 0$ which has multiplicity 5 at $(0,0)$. In characteristic zero, HH_2 and HH_3 are non-isomorphic A-modules, so the same is true for generic finite characteristic. A similar calculation shows that HH_2 and HH_3 are non-isomorphic for the rational curve

$$y^4 - 2x^3y^2 - 4x^4y - x^5 + x^6 = 0$$

uniformized by $x = t^4$, $y = t^5 + t^6$, as long as the characteristic is not 2 or 11. The latter curve has only a quartic singularity at $(0,0)$. It would be interesting to know if there are any cubic examples; as we shall see, there are no quadratic ones.

2. The topological Hochschild homology sheaf

We will use Bökstedt's definition [3] of the topological Hochschild homology spectrum THH(A) of a unital ring A. See also [18] for the product structure which exists when A is commutative, and [23] for a more detailed explanation of the notation which will be used.

For I (the skeleton of the) category of finite sets and inclusions, we look at cartesian powers I^m, and define

$$T_r^{(m)} = \underset{(j_0,j_1,\ldots,j_r)\in(I^m)^{r+1}}{\underrightarrow{\mathrm{holim}}} (S^{j_0}\wedge S^{j_1}\wedge\cdots\wedge S^{j_r}, A[S^{j_0}]\wedge A[S^{j_1}]\wedge\cdots\wedge A[S^{j_r}]\wedge S^m),$$

where for each $i\in I$, S^i is a wedge product of i copies of S^1, and for each $j=(i_1,\ldots,i_m)\in I^m$, S^j is the smash product of the S^{i_a}. (The indexing by I^m rather than by I is for the definition of the product when A is commutative, but topologically S^j is just a sphere, of dimension equal to the sum of cardinalities of its coordinates).

For each m, $T_\cdot^{(m)}$ forms a simplicial set with Hochschild-type face and degeneracy maps. For each r, $T_r^{(m)}$ is a spectrum equivalent to the smash product of the Eilenberg–Mac Lane spectrum HA of A with itself $r+1$ times.

Then $\mathrm{THH}^{(m)}(A)$ is the simplicial realization

$$\mathrm{THH}^{(m)}(A) = \coprod_r \mathrm{THH}^{(m)}(A)\times\delta^r/\sim,$$

which has spectrum stabilization maps since the stabilization maps of the $T_r^{(m)}$ commute with their simplicial structure maps. The resulting spectrum is called THH(A).

Similarly, we can define

$$\mathrm{THH}^{(m)\le i}(A) = \coprod_{r\le i} \mathrm{THH}^{(m)}(A)\times\delta^r/\sim,$$

the i-skeleton of $\mathrm{THH}^{(m)}(A)$, and use the spectrum stabilization maps to obtain a spectrum we will call $\mathrm{THH}^{\le i}(A)$.

In Bökstedt's first calculations of topological Hochschild homology of $\mathbb{Z}$ and $\mathbb{F}_p$ in [4], he made the following observation:

Lemma 2.1. *The topological Hochschild homology spectrum of a unital ring A is always a product of Eilenberg–Mac Lane spectra.*

Proof. This is because the obvious stabilization and multiplication into the zeroth coordinate allow us to exhibit THH(A) as a retract of $H\mathbb{Z}\wedge\mathrm{THH}(A)$; see [4] or [23]. □

Bökstedt uses the filtration by simplicial skeleta to get a spectral sequence calculating the stable homology of THH, from which he deduces the stable homotopy. The same filtration gives a spectral sequence calculating the stable homotopy of the Ω-spectrum

THH directly. Using the notation $E^k_{*,*}(\mathrm{THH}(A))$ to denote the E^k term of this spectral sequence, we have

$$E^1_{r,*}(\mathrm{THH}(A)) = \pi^S_*(HA^{\wedge(r+1)}).$$

Since $\pi^S_*(HA \wedge X) = \tilde{H}_*(X;A)$, $r+1$ applications of the universal coefficient theorem give

$$\pi^S_*(HA^{\wedge(r+1)}) \xleftarrow{\sim} \pi^S_*(H\mathbb{F}_p^{\wedge(r+1)}) \otimes A^{\otimes(r+1)}$$

and thus

$$E^1_{r,*}(\mathrm{THH}(A)) \xleftarrow{\sim} A^{\otimes(r+1)} \otimes \pi^S_*(H\mathbb{F}_p^{\wedge(r+1)}), \tag{2.1}$$

with d^1 preserving the tensor factors and acting as the Hochschild homology differential on the first factor. The second factor is the image under the unit map of $E^1_{r,*}(\mathrm{THH}(\mathbb{F}_p))$. We thus obtain

$$E^2_{*,*}(\mathrm{THH}(A)) \xleftarrow{\sim} \mathrm{HH}_*(A) \otimes E^2_{*,*}(\mathrm{THH}(\mathbb{F}_p)), \tag{2.2}$$

where each $\mathrm{HH}_i(A)$ sits in bidegree $(i,0)$ in the double complex.

Note that decomposition (2.2) is natural in A.

Since the spectrum multiplication respects the filtration by skeleta, the spectral sequence (2.2) is a spectral sequence of algebras.

Claim 2.2. *For A commutative*, $\mathrm{THH}_*(A)$ *is a ring.*

Proof. The ring structure is that induced on homotopy groups by the space-level product of Section 1.7 of [18]. □

Lemma 2.3. *For A commutative*, $\mathrm{THH}_*(A)$ *is a graded A-algebra, and in particular: has exponent p.*

Proof. This follows trivially from Claim 2.2, since $A = \mathrm{THH}_0(A)$ and A is an $\mathbb{F}_p$-algebra. □

Definition 2.4. We say that THH *splits weakly* for an $\mathbb{F}_p$-algebra A if the decomposition of (2.2) lasts till the E^∞-term, i.e. all the spectral sequence differentials vanish on the $\mathrm{HH}_*(A) \otimes 1$ terms to give us a canonical isomorphism

$$E^\infty_{*,*}(\mathrm{THH}(A)) \xleftarrow{\sim} \mathrm{HH}_*(A) \otimes E^\infty_{*,*}(\mathrm{THH}(\mathbb{F}_p)),$$

where $1 \otimes E^\infty_{*,*}(\mathrm{THH}(\mathbb{F}_p))$ is the image of the corresponding E^∞ term for $\mathrm{THH}(\mathbb{F}_p)$; and moreover the extension data is trivial: if we let

$$\mathrm{Fil}_i(\mathrm{THH}_*(A)) = \mathrm{im}(\mathrm{THH}^{\le i}_*(A) \to \mathrm{THH}_*(A)),$$

the short exact sequences of A-modules

$$0 \to \mathrm{Fil}_{i-1}(\mathrm{THH}_{i+j}(A)) \to \mathrm{Fil}_i(\mathrm{THH}_{i+j}(A)) \to E^{\infty}_{i,j}(\mathrm{THH}(A)) \to 0 \tag{2.3}$$

are all (non-canonically) split.

Note that if THH splits weakly for a commutative $\mathbb{F}_p$-algebra A, we have an A-module isomorphism

$$\mathrm{THH}_*(A) \cong \mathrm{HH}_*(A) \otimes \mathrm{THH}_*(\mathbb{F}_p). \tag{2.4}$$

Lemma 2.5. *If B is an étale A-algebra, then there is a natural isomorphism*

$$\mathrm{THH}_*(A) \otimes_A B \xrightarrow{\sim} \mathrm{THH}_*(B). \tag{2.5}$$

Proof. See [17, Lemma 2.4.2]. □

Lemma 2.6. *If $\{A_\alpha\}_\alpha$ is a filtered system of rings with direct limit A,*

$$\varinjlim_\alpha \mathrm{THH}_*(A_\alpha) \xrightarrow{\sim} \mathrm{THH}_*(A).$$

Proof. Clearly on the E^1-terms (2.1),

$$\varinjlim_\alpha E^1_{*,*}(\mathrm{THH}(A_\alpha)) \xrightarrow{\sim} E^1_{*,*}(\mathrm{THH}(A)).$$

Since differentials commute with limits, this continues by induction to the E^k-terms for all k, and so to the E^∞-terms. The result for THH then follows by the five-lemma. □

Proposition 2.7. *If B is an étale A-algebra and* THH *splits weakly for A, then it splits weakly for B. If B is a faithfully flat étale A-algebra, then* THH *splits weakly for A if and only if it splits weakly for B.*

Proof. $\mathrm{HH}_*(B)$ is generated by the image of $\mathrm{HH}_*(A)$ and by $B = \mathrm{HH}_0(B)$. All spectral sequence differentials vanish on the image of $\mathrm{HH}_*(A)$ by naturality, and on $\mathrm{HH}_0(B)$ since we have a first-quadrant spectral sequence. So by the multiplicativity of the spectral sequence we get that the requisite differentials vanish on $\mathrm{HH}_*(B)$.

To show the splitting of the sequences

$$0 \to \mathrm{Fil}_{i-1}(\mathrm{THH}_{i+j}(B)) \to \mathrm{Fil}_i(\mathrm{THH}_{i+j}(B)) \to E^{\infty}_{i,j}(\mathrm{THH}(B)) \to 0, \tag{2.6}$$

we first use induction on j to prove that for all i and j,

$$\mathrm{Fil}_i(\mathrm{THH}_{i+j}(B)) \xleftarrow{\sim} \mathrm{Fil}_i(\mathrm{THH}_{i+j}(A)) \otimes_A B.$$

For $j = 0$, this follows from Lemma 2.5, since $\mathrm{Fil}_i(\mathrm{THH}_i(B)) = \mathrm{THH}_i(B)$ for any ring B.

By the vanishing of the differentials, we know that

$$E^{\infty}_{*,*}(\mathrm{THH}(B)) \cong B \otimes_A E^{\infty}_{*,*}(\mathrm{THH}(A)). \tag{2.7}$$

Now since the sequence (2.3) is exact for A and B is étale, the sequence

$$\begin{aligned} 0 \to B \otimes_A \mathrm{Fil}_{i-1}(\mathrm{THH}_{i+j}(A)) &\to B \otimes_A \mathrm{Fil}_i(\mathrm{THH}_{i+j}(A)) \\ &\to B \otimes_A E^{\infty}_{i,j}(\mathrm{THH}(A)) \to 0 \end{aligned} \tag{2.8}$$

is also exact. By (2.7), the obvious map of complexes from (2.8) to (2.6) is an isomorphism for the $E^{\infty}_{i,j}$ terms. The inductive step follows from an application of the five-lemma.

Now since the short exact sequence (2.3) splits for A, so does the short exact sequence (2.8), which we have just shown to be isomorphic to the short exact sequence (2.6).

If B is faithfully flat over A, a non-zero differential in the spectral sequence for A gives rise to a corresponding non-zero differential in the spectral sequence for B, and the splitting of (2.8) implies the splitting of (2.3). □

Lemma 2.8. *Let $\{A_\alpha\}_\alpha$ be a filtered system of rings with limit A such that whenever $\alpha \le \beta$, A_β is étale over A_α. Then if* THH *splits weakly for some A_{α_0},* THH *also splits weakly for A.*

Proof. The vanishing of the spectral sequence differentials for THH(A) follows from the proof of Lemma 2.6.

For the splitting of the short exact sequences (2.3), observe that $A = \varinjlim_{\alpha_0 \le \alpha} A_\alpha$. But, for $\alpha_0 \ge \alpha$, we know from the proof of Proposition 2.7 that the exact sequences (2.3) for A_α are canonically isomorphic to the respective sequences for A_{α_0}, tensored over A_{α_0} with A_α. Thus in the limit, the short exact sequences (2.3) for A are isomorphic to the respective (split) sequences for A_{α_0}, tensored over A_{α_0} with A. Thus the sequences for A must split, too. □

Lemma 2.9. *If A is a ring and S a multiplicative system in A, then there is a canonical isomorphism*

$$\mathrm{THH}_k(A) \otimes_A S^{-1}A \xrightarrow{\sim} \mathrm{THH}_k(S^{-1}A)$$

for all $k \in \mathbb{N}$. If THH *splits weakly for A, the same is true for $S^{-1}A$.*

Proof. As $S^{-1}A$ is always the limit of étale algebras over A, this is immediate from Lemmas 2.5, 2.6, and 2.8, and Proposition 2.7. □

Lemma 2.10. *If $\mathfrak{p}$ is a prime ideal of A and $\tilde{A}_{\mathfrak{p}}$ is the henselization of $A_{\mathfrak{p}}$, then there is a canonical isomorphism*

$$\mathrm{THH}_k(A) \otimes_A \tilde{A}_{\mathfrak{p}} \xrightarrow{\sim} \mathrm{THH}_k(\tilde{A}_{\mathfrak{p}})$$

for all $k \in \mathbb{N}$. If THH *splits weakly for A, the same is true for $\tilde{A}_{\mathfrak{p}}$.*

Proof. Again, $\tilde{A}_{\mathfrak{p}}$ is the direct limit of étale A-algebras. □

Theorem 2.11. *Let X be a variety over a finite field. For all $k \in \mathbb{N}$, the sheafification $\mathscr{T}\mathscr{H}\mathscr{H}_{k,X}$ of the presheaf*

$$U \mapsto \mathscr{T}\mathscr{H}\mathscr{H}_k(\Gamma(U, \mathcal{O}_X))$$

is a coherent sheaf of $\mathcal{O}_X$-modules.

Proof. The fact that $\mathscr{T}\mathscr{H}\mathscr{H}_{k,X}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules is immediate from Lemma 2.9. In fact, Lemma 2.5 shows that it is actually a sheaf in the étale topology. To see that it is coherent, observe from (2.2) and Lemma 1.1 that for A a finitely generated $\mathbb{F}_p$-algebra, $E^2_{i,j}(\mathrm{THH}(A))$ is a finitely generated A-module for any i and j. This means that higher terms $E^k_{i,j}$ of the spectral sequence, which are subquotients of the $E^2_{i,j}$-terms, will also be finitely generated, as will the $E^\infty_{i,j}$-terms. (A is Noetherian since it is finitely generated over $\mathbb{F}_p$.) A finite extension of coherent modules is again coherent. □

Remark 2.12. By functoriality and the universal property of sheaves, the Dennis trace map [5] extends to a morphism of sheaves from $\mathscr{K}_{k,X}$, the sheafification of the presheaf of K_k, to $\mathscr{T}\mathscr{H}\mathscr{H}_{k,X}$.

Lemma 2.13. *If Π is a finitely generated monoid with unit,* $\mathrm{THH}_*(\mathbb{F}_p[\Pi])$ *splits canonically.*

Proof. This follows trivially from Section 6.1 in [18], where it is shown that

$$\mathrm{THH}(\mathbb{F}_p[\Pi]) \simeq |N^{\mathrm{cy}}_{\cdot}(\Pi)| \times \mathrm{THH}(\mathbb{F}_p),$$

where $N^{\mathrm{cy}}_{\cdot}(\Pi)$ is the cyclic bar construction on the monoid Π. This implies that

$$\begin{aligned} H^S_*(\mathrm{THH}(\mathbb{F}_p[\Pi]); \mathbb{F}_p) &\xleftarrow{\sim} H_*(|N^{\mathrm{cy}}_{\cdot}(\Pi)|) \otimes H^S_*(\mathrm{THH}(\mathbb{F}_p); \mathbb{F}_p) \\ &= \mathrm{HH}_*(\mathbb{F}_p[\Pi]) \otimes H^S_*(\mathrm{THH}(\mathbb{F}_p); \mathbb{F}_p), \end{aligned} \tag{2.9}$$

as modules over the dual $\mathscr{A}$ of the mod p Steenrod algebra.

By Lemma 2.1, $\mathrm{THH}(\mathbb{F}_p[\Pi])$ is a product of Eilenberg–Mac Lane spectra. By Lemma 2.3, all these Eilenberg–Mac Lane spectra correspond to groups with exponent p. So in fact $\mathrm{THH}(\mathbb{F}_p[\Pi])$ is a product of Eilenberg–Mac Lane spectra of $\mathbb{Z}/p\mathbb{Z}$, and its stable homology is a tensor product of its stable homotopy with $\mathscr{A} = H^S_*(H\mathbb{Z}/p\mathbb{Z}; \mathbb{F}_p)$.

To recover $\mathrm{THH}_*(\mathbb{F}_p[\Pi])$ from $H^S_*(\mathrm{THH}(\mathbb{F}_p[\Pi]);\mathbb{F}_p)$, then, apply the augmentation map $\mathscr{A}\to\mathbb{F}_p$ to the free $\mathscr{A}$-module $H^S_*(\mathrm{THH}(\mathbb{F}_p[\Pi]);\mathbb{F}_p)$, obtaining

$$\mathrm{THH}_*(\mathbb{F}_p[\Pi]) \xleftarrow{\sim} \mathrm{HH}_*(\mathbb{F}_p[\Pi]) \otimes \mathrm{THH}_*(\mathbb{F}_p)$$

as algebras, and in particular as $\mathbb{F}_p[\Pi]$-modules.

It is easy to trace the $\mathrm{HH}_*(\mathbb{F}_p[\Pi])\otimes 1$ factor and see that it lands in the corresponding factor in (2.2). □

Proposition 2.14. *If* THH *splits weakly for the henselization of every local ring* $A_\mathfrak{p}$ *of* A, *then* THH *splits weakly for* A.

Proof. By [14, IV, 18.6.6 (iii)], $\tilde{A}_\mathfrak{p}$ is faithfully flat over $A_\mathfrak{p}$ (which is flat over A). Therefore, tensoring with $\tilde{A}_\mathfrak{p}$ commutes with homology for any complex of A-modules and, moreover, for every non-trivial A-module M, there exists $\mathfrak{p}$ such that $M\otimes\widetilde{A}_\mathfrak{p}$ is non-trivial. The E_2-term of the spectral sequence of $\widetilde{A}_\mathfrak{p}$ is obtained from the E_2-term for A by tensoring. It follows that the same thing is true for every E_r, and if any differential were non-trivial on a class in $\mathrm{HH}_*(A)\otimes 1$, the same would be true for the corresponding differential in some $\mathrm{HH}_*(\widetilde{A}_\mathfrak{p})\otimes 1$, contrary to hypothesis. Likewise, if any extension class were non-trivial for A, it would be remain non-trivial after tensoring with some $\widetilde{A}_\mathfrak{p}$. Note that this latter statement is true only for affine varieties; this keeps us from globalizing to general varieties. □

Proposition 2.15. *If* Π *is a finitely generated monoid with unit and* $K/\mathbb{F}_p$ *a finitely generated field over* $\mathbb{F}_p$, *then* $\mathrm{THH}_*(K[\Pi])$ *splits weakly.*

Proof. As $\mathbb{F}_p$ is perfect, K is separably generated over $\mathbb{F}_p$. Let $F\subset K$ be a subfield such that $F=\mathbb{F}_p(\pi_1,\dots,\pi_n)$ is purely transcendental over $\mathbb{F}_p$ and K is separable over F. Let $\Pi'=\Pi\times\mathbb{N}^n$, where $\mathbb{N}^n$ is the free monoid generated by $x_1,\dots,x_n$. Then

$$\mathbb{F}_p[\Pi']\cong\mathbb{F}_p[\pi_1,\dots,\pi_n][\Pi],$$

so $F[\Pi]$ is a localization of $\mathbb{F}_p[\Pi']$, for which THH splits by Lemma 2.13, and $K[\Pi]$ is a direct limit of étale localizations of $F[\Pi]$. □

3. Monoidal singularities

Let Π be a finitely generated commutative monoid. We say Π is *graded* if there exists a homomorphism $\phi:\Pi\to\mathbb{N}$ such that $\phi^{-1}(0)=\{0\}$. Given a field K and such a monoid, we have a natural graded ring $K[\Pi]$ with augmentation ideal $I=K[\Pi\backslash\{0\}]$.

Definition 3.1. A prime ideal $\mathfrak{p}$ of an algebra A is *monoidal* if $\hat{A}_\mathfrak{p}\cong\widehat{K[\Pi]}$ where K is the residue field of A at $\mathfrak{p}$ and Π is a graded finitely generated commutative monoid.

Lemma 3.2. *Every non-singular point of an affine variety is monoidal.*

Proof. Let $\Pi = \mathbb{N}^{\mathrm{ht}(x)}$. Then $K[\Pi]$ is a polynomial ring in $\mathrm{ht}(x)$ variables, so its completion is a power series ring in the same number of variables. □

Lemma 3.3. *Let $U \subset X$ be an affine open subvariety of a toroidal embedding in the sense of* [20] *defined over $\mathbb{F}_p$. Then every point of X is monoidal.*

Proof. The question is formal, so it suffices to consider a point on an affine toric variety, that is, the spectrum of $\mathbb{F}_p[\Pi]$ where Π is the set of integral points of a rational polyhedral cone in $\mathbb{R}^n$, and $\Pi \cap -\Pi = \{0\}$. A rational hyperplane which meets Π only at 0 defines a grading of Π up to normalization, and it follows that $\mathbb{F}_p[\Pi]$ is monoidal. □

Theorem 3.4. *If every prime ideal of A is monoidal, then*

$$\mathrm{THH}_k(A) \cong \mathrm{HH}_k(A) \oplus \mathrm{HH}_{k-2}(A) \oplus \mathrm{HH}_{k-4}(A) \oplus \cdots$$

as A-modules.

Proof. By Proposition 2.14, it is enough to prove that THH splits weakly for every henselization $\tilde{A}_{\mathfrak{p}}$. As $A_{\mathfrak{p}}$ is monoidal, $\hat{A}_{\mathfrak{p}}$ is isomorphic to $\widehat{K[\Pi]}$ for some field K and gradable monoid Π. By [2], $\tilde{A}_{\mathfrak{p}}$ is isomorphic to $\widetilde{K[\Pi]}$. By Lemmas 2.10 and 2.15, THH splits weakly for $\tilde{A}_{\mathfrak{p}}$. □

Corollary 3.5. *If $P(x,y) \in \mathbb{F}[x,y]$ such that $A = \mathbb{F}[x,y]/(P(x,y))$ is reduced and every prime ideal of A is monoidal, then*

$$\mathrm{THH}_k(A) = \begin{cases} A \oplus (A/(\bar{P}_x, \bar{P}_y))^{k/2} & \text{if } k \text{ is even,} \\ A^2/\{(P_x a, P_y a) \mid a \in A\} \oplus (A/(\bar{P}_x, \bar{P}_y))^{(k-1)/2} & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Immediate from Theorem 3.4 and Proposition 1.7. □

Remark 3.6. By Lemma 2.5,

$$\mathrm{THH}_k(A) \otimes \mathbb{F}_q \xrightarrow{\sim} \mathrm{THH}_k(A \otimes \mathbb{F}_q).$$

Therefore, $\mathrm{THH}_k(A)$ is determined by $\mathrm{THH}_k(A \otimes \mathbb{F}_q)$ for any $\mathbb{F}_q$ over $\mathbb{F}_p$. Of course, singularities of $\operatorname{Spec} A$ which are non-monoidal may become monoidal after an extension of scalars. In particular, by Proposition 2.7, Theorem 3.4 continues to hold under the weaker hypothesis that A is *potentially monoidal.*

The rest of this section is devoted to additional examples of monoidal singularities. By the above remark, we are justified in working over $\mathbb{F} = \bar{\mathbb{F}}_p$.

Example 3.7. The point $\operatorname{Spec} \mathbb{F}[x]/(x^n)$ is monoidal by definition. This was the motivating example. See [23].

Example 3.8. An ordinary double point on a plane curve is monoidal, since such a singularity is well-known to be analytically isomorphic to $\mathbb{F}[x,y]/(xy)$.

Example 3.9. An ordinary triple point of a plane curve is monoidal as it is analytically isomorphic to the singularity of $\mathbb{F}[x,y]/(x^2y-y^2x)$ by [16, I, Example 5.14c]

Example 3.10. If $p>2$, all double points are monoidal. Indeed, by [16, I, Example 5.14d], every double point is analytically isomorphic to the singularity of $\mathbb{F}[x,y]/(y^2-x^n)$ for some value of n. The cases of node, cusp, and tacnode correspond to $n=2$, $n=3$, and $n=4$, respectively. Note that, of course, we assign x degree 2 and y degree n in the gradation.

Example 3.11. Quadruple points need not be monoidal, as Example 1.8 shows.

Example 3.12. Double points on a curve in characteristic $p>2$ are examples of *simple* curve singularities. In general, a simple singularity on a variety of fixed dimension over an algebraically closed field of fixed characteristic p is determined up to analytic isomorphism by a simply laced root system, together with some additional data for small values of p. The normal forms are worked out explicitly in [21]. For $p>2$, double points are the singularities of type A_n. Again for $p>2$, the singularities of type D_n are analytically isomorphic to $\mathbb{F}[x,y]/(x^2y-y^{n+1})$. For $p>3$, the singularities of type E_6 and E_7 are given by $\mathbb{F}[x,y]/(y^3-x^4)$ and $\mathbb{F}[x,y]/(x^3-xy^3)$, respectively, while for $p>5$, the singularities of type E_8 are analytically isomorphic to $\mathbb{F}[x,y]/(y^3-x^5)$. In every case, the singularities are monoidal.

Example 3.13. Another generalization of Example 3.8 is that of semi-normal singularities of curves. Over an algebraically closed field, such singularities are analytically isomorphic to the union of coordinate axes in n-space, that is, to

$$\mathbb{F}[x_1,\dots,x_n]/(x_1x_2,x_1x_3,\dots,x_1x_n,x_2x_3,\dots,x_{n-1}x_n).$$

Thus semi-normal singularities are monoidal. The Hochschild homology of the local rings of such singularities is computed in [11] in characteristic zero, but the result does not depend on characteristic.

Example 3.14. Simple singularities of surfaces need not be monoidal even for $p\gg 0$. However, surface singularities of type A_n are always monoidal [13]. In particular, ordinary conic points of a surface are monoidal. The Hochschild homology of such singularities can in principle be calculated by the formula of [15, Eq. (3.5)].

Example 3.15. In any dimension and any characteristic, singularities which are analytically isomorphic to those of a divisor with normal crossings are monoidal. Again, the homology can be computed from [15, Eq. (3.5)].

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